The Renaissance of Numbers: on Continuity, Nature of Complex Numbers and the Symbolic Turn

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ABSTRACT
The paper presents an analysis of imaginary quantity before Gauss based on the notion of continuity and symbolic representation. Its aim is to uncover subtle roots of the “impossible”, “sophisticated” or “absurd” entities that, as we claim, stem from the Renaissance notion of nature and from the “symbolic turn” which occurred in that period. In order to grant impossible quantities a reasonable (operational) meaning, it is necessary to establish an equation (formal continuity) between real and imaginary. It is possible only if the real is in a sense subsumed within the symbolic which holds paradigmatically for the notions of number and magnitude. For, once number and magnitude become symbolic representations of the same universal intellectual matter of quantity, an operational analogy and continuity between them may be established. Three “continuities” shall be distinguished on the path to such “universal mathematics” at the end of which the imaginary entities may acquire the citizenship in the Republic of numbers.
...sic tamen operabimur¹

**SERIES OF SERIES.**

Complex numbers² represent a solution to any polynomial equation. Thus, a complex number designates an algebraic solution in general and illustrates among other things the universal validity of the fundamental theorem of algebra:³ Thanks to complex numbers every equation has a solution, and consequently the number of solutions it has corresponds to the degree of the equation. The universality which eventually turned out to be the enclosure of the complex field⁴ has indeed been

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² A complex number refers to an entity expressed in the form \( a + bi \), where \( a, b \in \mathbb{R} \), while \( i \) represents a solution to the equation \( i^2 = -1 \). Given that no real number satisfies such a relation, \( b \) is called an “imaginary” component of the complex number, whereas \( a \) is a “real” part. Complex numbers establish a field \( \mathbb{C} \) which, by definition, contains the field \( \mathbb{R} \) inasmuch as \( b \) allows the value 0. Consequently, complex numbers constitute an extension, namely algebraic enclosure, of the real numbers, are subject to analogical operations as the real numbers, bear analogical properties (primality, for example) and enjoy the laws of real addition, multiplication and commutativity.
³ See Girard 1629, E4 r–v; AT VI, pp. 444–454; Newton 1707, pp. 252–255; d’Alembert 1746, pp. 182–224; Euler 1751; Gauss 1799. There are countless works on the fundamental theorem, for a brief historical overview see Katz, & Parshall 2014, pp. 247–288.
⁴ No other kind of complex quantities or numbers than that of the form \( a + bi \) is actually necessary for the solution of
the driving force behind these suspicious and disturbing objects. Impossible, sophisticated, absurd, imaginary as they were called, these “amphibians between being and non-being”⁵ are nonetheless wonderfully well suited to reflect the “treasure of relations between real quantities”,⁶ indicating the subtle continuity, inherent ambiguity and symbolic nature of things characteristic for the Renaissance period.

How can a mere “game with empty symbols”⁷ possess such expressive power? By what right can these enigmatic entities acquire citizenship in the Republic of numbers? One of the remarkable properties of numbers consists in symbolizing an order—and the other way round, a number may be defined as an element of an order. Now one of the simplest kinds of order is the order of succession and this is how numbers can be defined: as the elements of an infinite sequence or “series” (Reihe) allowing a “relation or transition” between the elements, defining their rank and relative distance. Thus, for instance, if the relation between A and B is 1, then the opposite relation, between B and A, is −1. Given an arbitrary starting point, apparently, every integer symbolizes a relation between the starting point and some element of the infinite sequence.⁸ At the end of the day, it is precisely the system of these relations which constitutes the proper object of the science of arithmetic.

The above reasoning, despite its apparent simplicity, marks an achievement of mathematical abstraction. Not only does it provide us quite a reasonable way of conceive of the notion of negative quantity (for how could something possibly be lesser than nothing?), but also and foremost—it erects a self-contained structure allowing for a recursive definition of ever more complex, yet ever more general objects. For we are able to conceive of objects of such a nature that they cannot be ordered except by the means of two sequences, or according to “sequences of sequences”. Besides the “original units” (vorigen Einheiten) +1 and −1, the list of transitions may be complemented by another pair of transitions, say +i and −i. Thus, a two-dimensional manifold (Mannigfaltigkeit) is formed, that comprises “transitions from one sequence to the other”, and consequently all the relations holding between the objects defined by them. Now, given that all that concerns a mathematician here is “enumeration and comparison of their relations”,⁹ we are free to represent them in corresponding special relations:

(...) and the simplest case is when there is no reason to arrange the symbols of the objects in any other way than quadratically. That is, one

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⁵ GM VI, p. 357.
⁶ Gauss 1876, p. 175.
⁷ “(...) an sich inhaltleeres Zeichenspiel” (Gauss 1876, p. 175). Among other epithets, we can read “blind”, “opaque” or “denatured”.
⁸ Gauss 1876, p. 176.
⁹ Gauss 1876, p. 176.
divides an unbounded plane into squares by means of two systems of parallel lines which cross each other at right angles, and one assigns the points of intersection to the symbols. Every such point \( A \) has four neighbours, and if one designates the relation of \( A \) to a neighbouring point by \(+1\), then the point to be designated by \( -1 \) is thereby determined, and one can choose either of the two others for \(+i\) or can take the point related to \(+i\) to be right or left as one wishes.\(^{10}\)

The purely symbolic, punctual character, emphasized by Gauss himself, of the newly constructed objects is obviously of the utmost importance here—and the same holds for the self-determination of the whole system, once one of the transitions has been specified as a positive number series. The universality of the scheme underlying the realms of both numbers and magnitudes grants us access to the decisive insight:

But that means, in the language of the mathematicians, that \(+i\) is a geometric mean (Proportionalgröße) between \(+1\) und \( -1 \) or corresponds to the sign \( \sqrt{-1} \): we intentionally do not say the geometric mean, for clearly \(-i\) has the same claim. Here, therefore, an intuitive (anschaulichen) meaning of \( \sqrt{-1} \) is completely established (gerechtfertigt), and one needs nothing further to admit this quantity into the domain of the objects of arithmetic.\(^{11}\)

Here we finally stand on the firm ground of the Gauss (Argand) plane.\(^{12}\) One of the most, in Gauss’s words, “obscure and mysterious” entities of the world of mathematics, magnitude \( \sqrt{-1} \), has all suddenly turned out to be a number entirely admissible in arithmetic\(^{13}\) and to shine a bright new light, explaining and unifying branches of mathematics as disparate as arithmetic, trigonometry or integral calculus might be.\(^{14}\) The stone that the builders rejected has become the cornerstone of analysis.\(^{15}\)

The relational symbolic standpoint taken above allowed us to grasp the more intimate, universal and manifold nature of familiar mathematical entities, since it provides a proper perspective in which the intuitive (geometric) and the formal (algebraic) comprehension explain each other. Accordingly, the key notion here, that of the geometric mean, \( i \) or \(-i\), is

\(^{10}\) Gauss 1876, p. 177; tr. Ewald 2005a, p. 312.

\(^{11}\) Gauss 1876, p. 177; tr. Ewald 2005a, p. 312.

\(^{12}\) As the name of the co-inventor indicates (see Argand 1806), the geometrical interpretation of imaginary quantities has, in spite of the overwhelming clarity and consistency of the Gaussian account, rather a complex history beginning at the very latest with Wallis’s Algebra (1673). It includes the names like H. D. Trudel, C. Wessel or A. Q. Bueé. See further, Flament 2003, pp. 97–248 and Granger 1998, pp. 63–73.

\(^{13}\) Gauss 1876, p. 177.

\(^{14}\) Up to present day, complex numbers find their place in the vast field of sciences varying from optics, statistics, computer science to quantum mechanics, including an occasional overlap to the sphere of religion and mysticism (see Florensky 1922).

\(^{15}\) See Psalm 118:22.
by no means simple. It is, rather, twofold. On the one hand it consists in the aforementioned relation given by the algebraic expression

\[
\frac{-i}{1} = \frac{1}{-i}
\]

but, on the other hand, it involves the meaning of the unitary geometrical transitions +1, −1, +i, −i representing rotations of an angle while, naturally, the rotation from +1 to i is equivalent to the one from i to −1.\(^\text{16}\) Since the rotation may be easily matched with a division of the angle, complex arithmetic metamorphosed into an essential analytical tool of trigonometry as well.\(^\text{17}\)

A device, as it were, for transferring relations from one mathematical discipline to another, complex arithmetic granted each of them universal access to the reasoning styles of the others, providing thus not only a common ground for unification of all fields of mathematics back then, but also, and for the same reason, a completely new horizon of conceiving of ever more complex and general objects, spaces and algebras. For, we can finally ask with Gauss himself whether and “why the relations between things that form a manifold of more than two dimensions cannot supply yet another type of number that is admissible in higher arithmetic”.\(^\text{18}\)

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**IMPOSSIBLE INTERSECTION**

Ever since its inception, European mathematics was marked by two consubstantial separations. The first one is the separation on the conceptual level between the continuous and the discrete. It emerged as a result of the discovery of incommensurability and paved the way to the notion of mathematical infinity. The second one, inherently related to the above, emerged on the level of doing, proving and reasoning. Here we may observe two basic mindsets or prevailing attitudes towards the matter of mathematics: demonstrating and calculating. The former, geometric by nature, pertains to the sphere of vision, intuition and intellection—it’s aim is to demonstrate the truth about the formal proprieties of figures by constructing new ones, inventing a common measure and stating conclusively some sort of equality. The latter is characterized by a combinatorial or algebraic approach. Rather than vision, it concerns tactile (blind),\(^\text{19}\) sequential manipulation of generalized objects that are usually endowed with different values or dignities. These tend to represent a symbolic value, or precisely a symbol of value—a value as

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16 Granger 1998, pp. 74–75. In the same manner we might recall Euler’s “proof” of the equality \(\sqrt{-2} \times \sqrt{-3} = \sqrt{6}\) intelligible only on condition that we let \(\sqrt{6}\) denote the pair ±\(\sqrt{6}\). See Stillwell 2010, pp. 280–281.

17 The first one to observe the equivalence between the famous problem of trisecting an angle and the solution of a cubic equation was François Viète in 1591; see Viète 1646, pp. 82–162.

18 Gauss 1876, p. 178; tr. Ewald 2005a, p. 313.

19 Cf. note 7. It is well known that the word “calculus” meant originally a pebble (in abacus, in a game or an election), a small smooth piece of limestone, which can actually bear very limited inner meaning (cf. Bovillus’s *pigur ut mineral*) and represent in itself any other nature whatsoever. In this sense, the “calculus” is completely blind.
such, given ultimately by the operation it is subject to. Rather than a figure, therefore, it is a configuration as a vehicle of relations that is at stake. The goal of such operational way of mathematics is to invent. Driven by the blind and formal power of rules, it consists in penetrating the horizons of complexity in order to disentangle an underlying meaning, take possession of it and use it—no matter if it is in the art of algebra, divination, medicine or kabbala.  

The tension that arises between these two vital strategies forms the history of European mathematics. Although there might have been intellectual periods where one seemingly dominated the other, both were continually present at each stage of development. They are both inextricably related and folded into each other, because they convey different symbolic expressions of the fundamental continuity of the “intellectual matter”. Both are ultimately based on the notion of analogy, and both, we claim, finally intersect in the notion of imaginary quantity, and complex numbers in particular.

Contrary to what the clarity of the Gaussian geometrical interpretation might suggest, a whole labyrinth of symbolic invention is concentrated in this enigmatic entity. Here, the “natural”, “intuitive” or “phenomenological” meanings of basic mathematical notions, like number and magnitude, are being constantly blurred within the context of their proper interrelations and reborn afterwards in the form of a code, as operational symbols or ciphers providing access to a more universal field of expression.

In this paper, we shall give a version of the history of imaginary quantities before Gauss, based on the notion of continuity and symbolic representation—as we have said, the Gaussian conception is in no way simple or “intuitive”. The goal of the paper is to uncover the most subtle roots of the imaginary quantity we find in the Renaissance notion of nature. We claim that it is on the basis of this notion that a “symbolic turn” could be introduced and an intelligible (operational) meaning conferred to an


21 On our view, the core of analogical reasoning lies in the definition of ratio (def. 3 and 4 of the Fifth book of Elements). For, every analogy presupposes some sort of continuity between the members of the analogy, while the geometrical magnitude represents the most characteristic expression or image of continuity. Ultimately, it is the geometrical continuity which stands analogically for the continuity between analogues of any other kind. But the notion of ratio—and that of the same ratio, i.e. proportion, especially—states the major structural property of the continuous geometrical magnitude: the mutual dependence between the measurability and the kind of magnitude. Consequently, the geometrical magnitude arises from the analogy which it founds in turn. Thus, it founds itself or, which means the same, arises without foundations, like everything that is truly continuous. See Makovský 2019, pp. 452–456.

22 The paradox of the complex simplicity behind the imaginary units may be illustrated in another sentence by Gauss: “(...) die höhere Arithmetik so oft Gelegenheit gibt, dass nich so wohl die Schönheit und Einfachheit der Theoreme selbst, als die Schwierigkeit ihrer Begründung sie vorzüglich merkwürdig macht”. (Gauss 1876, p. 167)
equation between real and imaginary. Once it is shown, we are going to focus on the intersection of the two aforementioned vital approaches that enabled the invention of the “impossible” quantity. We intend to indicate the way such an intersection could happen in relation with a general idea usually called universal mathematics.23 However, the very same conditions that allowed for the emergence of the “impossible” quantities gave rise to obstacles to any consistent or intuitive interpretation of them, and consequently to their “existence”, or acceptance among mathematical objects. These obstacles, as we shall see, arise from the “intuitive” comprehension of numbers. In order to overcome them, two “paths” of universal mathematics must converge and give rise, as we shall argue, to the symbolic nature of number. Consequently, the “naturalness” of imaginary entities can be established. An algebraic strategy of such an undertaking, an indication of an operational universality,24 will be presented in the conclusion of this paper—before we finally reach the Gaussian starting point.

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### A Subtle Quantity

The history of complex numbers is intertwined with that of negative ones. It begins at the moment, precisely, when a negative quantity appeared under the sign of square root. As a result of the “forbidden” operation an “impossible quantity” turns up.25 The new entity is therefore a purely theoretical product which is incommensurable, or rather utterly alien to a “real” quantity of any nature, but still stems from a real problem. More importantly, there are problems that can be solved only by means of the “absurd” operation. The impossible

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23 As the well-known Cartesian science “circa ordinem et mensuram” (AT X, pp. 377–378; cf. pp. 156–158), the universal mathematics seems indeed to exclude any notion of “imaginary” entity whatsoever. Although the situation is far more complex even in Descartes’s thought, this is perhaps one of the reasons—besides rather abstract and speculative nature—why universal mathematics and imaginary quantity has not been often put in relation in the secondary literature. To keep our account as transparent as possible we take the idea of universal mathematics in its most general sense: as a science aiming to overcome the separation between continuous and discrete. We draw consequences of this attempt in order to clarify the nature of the second separation, that between geometric and algorithmic mathematics, and thus indicate the conditions of possibility of imaginary quantity. Thus, the ambition of this essay is not to describe the historical phenomenon of mathesis universalis. Although it is true that it takes some inspiration from the “symbolic number” thesis exposed for instance by Klein (1968; cf. Rose 1975, p. 143; Giaquinta 2010, pp. 176–177), it also takes into account the influence of practical mathematics as well as the essential diversity and non-linearity of approaches towards universal mathematics (Rabouin 2009). The primary focus of this essay is conceptual; it is the universality and “richly ambivalent nature” (Rose 1975, p. 6) of Renaissance mathematics, insofar they stem from the notion of infinite and give rise to “higher mathematics” (Kästner 1758, p.5).

24 See note 4.

25 Chuquet 1880; see also Flegg, Hay, & Moss 1985, pp. 359–360. Prodromes of the new entity are to be found among ancient and non-European mathematicians (Flament 2003, p. 9–10).
quantity becomes unavoidable in order to obtain the real one.\textsuperscript{26}

The problem in question is the algorithmic extraction of the roots of the cubic equation: “Let the cube of $GH$ and six times the side $GH$ be equal to 20.”\textsuperscript{27} This means, in our notation, a simple equation $x^3 + 6x = 20$, and generally an equation of the type $x^3 + px = q$ having a real solution. Following a geometrically constructed rule today known as Cardano’s formula\textsuperscript{28}

\[ x = \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}} + \sqrt[3]{\frac{q^2}{4} - \frac{p^3}{27} + q} - \frac{p}{2} \]

we arrive at

\[ x = \frac{\sqrt[3]{108} + 10}{2} - \frac{\sqrt[3]{108} - 10}{2} \]

which yields the solution $2$ “as is perfectly clear if it is tried out.”\textsuperscript{29} Indeed, the root 2 can be easily unmasked by a mere inspection of the equation—but only under the assumption that $p$ is positive.

Geometrically speaking, this is always the case. But that is also the reason why the subsequent investigation into cubic equations split up into many subcases according to the disposition of the cubes, squares and lines in question. That’s why, in fact, Cardano’s formula applies uniquely to equations having one real solution. Discussion of the rules for these special cases then occupies the following series of chapters\textsuperscript{30} of the \textit{Ars magna}. While the positive or “true” solutions do not pose any problem here, since they are justified by relations between geometrical figures, sooner or later it appears inevitable to tackle negative ones. Not only does Cardano admit that, calling them “debitum”, “fictitious” or “false”, but he develops even a full-fledged theory of true and false roots and irréductible” in \textit{Encyclopédie méthodique II}, pp. 736–738; cf. Flament 2003, pp. 15–18; for a recent overview of Cardano’s system of algebra, see Confalonieri 2015.

\textsuperscript{26} The original term \textit{res} (chose, cosa) is to be stressed here, for it constitutes the genuine principle of the science of algebra. Compared to numbers, the (unknown) “things” are indeterminate in their own right. Consequently, they form much more general domain of solutions and extend the meaning of arithmetical operations performed on them (cf. Farès 2018). In this way, the algebraical expressions designate a set of relations as well as an object. The thing, accordingly, may by the same token designate a quantity as well as, for example, a lack of quantity, i.e. a negative quantity (cf. AT VI, p. 372).

\textsuperscript{27} Cardano 1968, p. 96.

\textsuperscript{28} “Cube one-third the coefficient of $x$; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same (…)” (Cardano 1968, pp. 98–99). The well-known bitter history around the discovery of Ferro-Tartaglia-Ferrari-Cardano’s rule is given in detail for example in Giaquinta 2010, pp. 177–181 or Katz, & Parshall 2014, pp. 215–219. For the algebraic derivation of the formula see d’Alembert’s article “Cas

\textsuperscript{29} Cardano 1968, p. 100.

\textsuperscript{30} The titles of the chapters reflect the special cases. “Chapter XII: On the Cube Equal to the first power and number” ($x^3 = px + q$); “Chapter XIII: On the Cube and Number Equal to the First Power” ($x^3 + q = px$); “Chapter XIX: On the Cube and Square Equal to the First Power and Number” ($x^3 + px^2 = rx + q$); etc.
the rules governing their multiplicities, symmetries and transformations.\textsuperscript{31}

The first important, if uncertain step \textit{ultra Herculis columnas} is executed. For, these rules lead, almost by formal necessity, to the consideration of the “impossible” case of the equation \(x(10 - x) = 40\), where the discriminant is negative—or, in Cardano’s terms, where we are forced to consider the root of a fictitious square:

The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced (…), leaving a remainder of \(-15\), the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be \(5 + \sqrt{-15}\) and \(5 - \sqrt{-15}\).\textsuperscript{32}

What is the \textit{nature} of such an enigmatic, ambiguous quantity? The side of a square of an owed land, after all, has to be positive; for, being negative the area would not be owed. Such a quantity floats on the boundary between positive and negative, real and fictitious and has something “sophisticated” to its nature, “since with it one cannot carry out the operations one can in the case of pure negative (\textit{puro minus}) and other [numbers].”\textsuperscript{33} It cannot be explained by a kind of negative assumption. An attempt at a geometric justification fails as well, since “a surface is far from the nature of a number and from that of a line, though somewhat closer to the latter.”\textsuperscript{34}

Despite all the odds, a subtle regularity of which the new entity is a result or manifestation could not pass by Cardano unnoticed. While the result is deemed “as refined (\textit{subtile}) as it is useless”,\textsuperscript{35} having the rule we possess, though, sufficient mathematical liberty to “put aside the mental tortures involved”\textsuperscript{36} and to proceed to foolish calculations, permitted by nature even less than dimensions higher than that of cube.\textsuperscript{37} Although, as we have said, the geometric justification by completing the square is impossible, the symbolic operation is surprisingly simple and natural and yields blindly the correct results which, most important, cannot be obtained in any other way. The spectre from the fictitious world, the

\begin{itemize}
\item \textsuperscript{31} Cardano 1968, pp. 10–11. Cardano’s example of the negatives is indeed illustrative: “The dowry of Francis’ wife is 100 \textit{aurei} more than Francis’ own property (…). We assume that Francis has \(-x\); therefore, the dowry of his wife is 100 – \(x\)” (Cardano 1663, p. 218).
\item \textsuperscript{32} Cardano 1968, p. 219.
\item \textsuperscript{33} Cardano 1968, p. 220.
\item \textsuperscript{35} “(…) \textit{Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile}” (Cardano 1663, p. 287b; Cardano 1968, p. 220).
\item \textsuperscript{36} “\textit{dimissis incrucationibus}”. Another rendering of the (deliberately?) ambiguous expression can be found—more technical “disregarding cross products”. See Gonzáles-Velasco 2011, p. 148.
\item \textsuperscript{37} Cardano 1968, p. 9.
\end{itemize}
impossible quantity, finally gains a foothold in the realm of intuitive clarity and exactness, even though for a short while, before vanishing during the calculation.

**SYMBOLIC QUANTITY AND UNIVERSAL MATHEMATICS**

What might have possibly led Cardano to imagine the impossible quantity of *radix minus* and grant it a place within the calculation? And what did prevent him, unlike his successor Rafael Bombelli, to analyse more carefully the rules of their conduct in order to tear down the masks of the real values of the roots? The answer to both questions is the same: the symbolic nature of the negative quantity. Contrary to his coevals, who simply—and, in one view, correctly—dismissed solutions involving negative numbers as impossible, Cardano acknowledged them. False, fictitious, non-existent as they were, they were present at least symbolically, as a result of an operation extended into a new domain of its generality; and we have seen that it was due to the symbolic presence of the negatives that Cardano was able to disentangle a subtler regularity among the roots of an equation, and reveal a hidden nature of quantity.

If the formal necessity of the calculation urged Cardano—*sponte sua* and as if against his will—to invite the false and fictitious into the solution of a true problem, the geometrical intuition, on the other hand, chased the symbolic creatures of the negative and imaginary quantity from the fundamental conditions of the problem itself. After all, Cardano himself put his thumb on the ambivalence discussing the question of negative powers, but the limitation holds generally for terms of all equations:

> “ideo imaginaberis R. ſ. 15” (Cardano 1663, p. 287b).

> “Sel numero qual si trova i la ditta equatone accompagnato con lo cen. sei no e minore: overamente equale al quadrato de la ½ de le cose el caso essere insolubile” (Pacioli 1523, p. 147a). This stems from basic assumption of geometric algebra: “(...) cosa ene ipossibile che piu cose se aguaglino a manco cose. Ne che ancora manco cose se aguaglino a piu cose et sic in alis.” (Pacioli 1523, p. 145a)

It is always presumed in this case, of course, that the number to which the power is equated is true and not fictitious. To doubt this would be as silly as to doubt the fundamental rule itself for, though opposite reasoning must be observed in opposite cases, the reasoning is still the same.

Despite the admitted symmetry of the opposite instances, a formal continuity between them is not allowed: negative quantities cannot figure as terms in an equation. It is forbidden by the nature of terms to equate *cose* to false numbers, since “what we call *debitum*, cannot be produced by any expansion of a true number.” Even if they happen to intersect somehow in boundary situations, the true and false (fictitious and

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38 “ideo imaginaberis R. ſ. 15” (Cardano 1663, p. 287b).

39 For instance, the most influential one, Luca Pacioli who in “Notandum utilissimum” of his *Summa de arithmetica* states clearly the discriminant condition for real solution: “Sel numero qual si trova i la ditta equatone accompagnato con lo cen. sei no e minore: overamente equale al quadrato de la ½ de le cose el caso essere insolubile” (Pacioli 1523, p. 147a). This stems from basic assumption of geometric algebra: “(...) cosa ene ipossibile che piu cose se aguaglino a manco cose. Ne che ancora manco cose se aguaglino a piu cose et sic in alis.” (Pacioli 1523, p. 145a)

40 See note 31.

41 Cardano 1968, p. 11.

42 See note 30.

43 Cardano 1968, p. 10.
real, possible and impossible) belong to heterogeneous orders of expression, and cannot therefore be taken into the same account.

And yet, this has to be done, at least for the sake of the universality and simplicity of the rules of algebra. How can such a decisive step be accomplished? Apparently, it is necessary to cross the imaginary borderline and establish a perfect continuity between natural and symbolic beings. In the end, such a perfect continuity guarantees that complex numbers observe the same rules of conduct under the sign of square root as real ones and may be subjected to analogous operations. But in the same manner as in reality, numbers and magnitudes hold even in the middle of a dream like “engraved on the back of copper plates”. Since things can be numbered and measured even in an utterly imaginary world, the only mathematical way to proceed is to introduce a symbolic turn: to make nature itself part of a more subtle and general symbolic reality and open the door to universal mathematics—or, which means the same, to an entire mathematization of the universe. It is a commonplace that this major step was effected in the times of the Renaissance. But, at the same time, a no less important transformation had to happen within the notion of quantity in order for the “false number” to be promoted on par with the true one, or even with none. It has to become symbolic first.

**MATHESIS UNIVERSALIS**
The Aristotelian canon of science separated the concepts of number and

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44 This kind of continuity is also the reason that “the complex quantities are not opposed to the real but contain them as a special case”, as Gauss pointed out, and “that theory appears in an entirely new light, and its results acquire a startling simplicity” (tr. Ewald 2005a, p. 309), cf. n. 2. We are about to see why.

45 “(...) *ad scribendum hortatus sum, cuius historiam a tergo aeararum nostrarum imaginum insculpi feci*” (Cardano 2013, p. xxi). It is known, according to Cardano’s own presentation, that his encyclopaedia of Renaissance natural philosophy, *De subtilitate* (see below, note 46), represented in reality (including its central concept—“matter that covers everything”—and its name) a continuous record of a series of the author’s dreams. Within this book of nature, geometry occupies its central part.

46 The term *subtilitas* (see note 35) has a special meaning here. “It is the feature (*ratio*) by which things that can be sensed are grasped with difficulty by the senses and things that can be understood are grasped with difficulty by the intellect” (Cardano 2013, p. 15). Thus, subtlety is a universal concept of natural philosophy pointing out to the boundary character of things which appear especially on the boundary between senses and understanding. The subtlety of things cannot be grasped by either way alone, but still is indispensable to reflect the unity of nature. This imaginary labyrinth in between the contraries constitutes the situation of the man of the Renaissance. As the “knot of the Universe”, the man, ultimately, is the key to the continuity of ambivalent symbolic nature, and consequently the Ariadne’s thread of the labyrinth.

47 The period of Renaissance is defined here precisely with respect to what we called the symbolic turn and delimited by the names of Cusanus and Leibniz. See further Makovský 2019. The role of grammatical and rhetorical approaches towards the nature in the symbolic turn, the Renaissance “linguistic turn”, may only be mentioned here.
There is no kind (genus) comprising number and magnitude: numbers are not magnitudes—in the same manner as they are not, for instance, angles or volumes. This is very significant. Since a science demonstrates the attributes of a kind (that is assumed to be real) it is defined by, a scientific demonstration holds only within one kind and may not be used in a science of another kind:

One cannot, therefore, prove anything by crossing from another genus—e.g. something geometrical by arithmetic. For there are three things in demonstrations: one, what is being demonstrated, the conclusion (this is what belongs to some genus in itself); one, the axioms (axioms are the things on which the demonstration depends); third, the underlying genus of which the demonstration makes clear the attributes and what is accidental to it in itself. Now the things on which the demonstration depends may be the same; but of things whose genus is different—as arithmetic and geometry, one cannot apply arithmetical demonstrations to the accidentals of magnitudes, unless magnitudes are numbers.  

Magnitudes are not numbers, nor correspond perfectly to them—because of the existence of irrational ratios or magnitudes. The continuous and the discrete are heterogenous natures. Therefore, there can be no mutual predication between them. Transgression of the kind is impossible—a fact that, at the end of the day, rules out from the science of mathematics a notion as fundamental as measure.

No doubt, such a theoretical purity, non-mathematical in essence, would cause a serious harm to mathematical practice of any kind. Even though the fragmentation was ingenuously overcome by Eudoxus’s theory of proportion allowing for the comparison of relations between magnitudes of the same kind, the separation between the continuous and the discrete in geometry remains in existence, as evidenced by the separate treatment of proportions in the Elements. The roots of the incommensurability between arithmetic and geometry grow deeper, stemming from nothing less than the basic nature of number, the monad: a unit supposed to have a direct correspondence to an intuition of the counted object, “a unit is that by virtue of which each of the things that exist is called one”. As such, the unit as the principle of number, “multitude composed of units”, is by nature indivisible:

The one is indivisible just because the first of each class of things is indivisible. But it is not in the same way

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50 Elem. def. VII.2. The unit is at the same time the principle of measure, “for the one means the measure of some plurality, and number means a measured plurality and a plurality of measures. Thus, it is natural that one is not a number” (Met. 1088a3–6). Necessarily, as a measure the unit cannot be divided.
that every ‘one’ is indivisible, e.g. a foot and a unit; the latter is absolutely indivisible, while the former must be placed among things which are undivided in perception (…)—for doubtless every continuous thing is divisible. 51

Numbers are not magnitudes. They do not correspond to magnitudes, nor may be subsumed under Eudoxus’s theory of proportions. For, not only is every continuous thing divisible, but every continuous thing can also be divided in any ratio 52—conditio sine qua non of the science of geometry. Despite the very many apparent regularities, coincidences and common features, the separation between arithmetic and geometry is radical; according to Aristotle, the “common notions” (koine ennoia), “common propositions” holding both for “units and points and lines” are not common but “by analogy, since things are useful in so far as they bear on the genus of the science”. 53 They break down every time they touch the indivisible unit—almost as if one, following the greatness of magnitude, 54 crashes into the limit of heavens.

Greek analysis of continuity, for the reasons explained above, followed a different path of thought than that of number. Numbers, due to the correspondence between the unit and the object, thought to be one, were always natural numbers, i.e. positive integers, while the construction of ratios, ultimately ratios between integers, equivalent to today’s set of positive rational numbers, 55 was, precisely, a means to preserve the indivisibility of the unit and irreducibility of measure. As such, it was principally disconnected from the consideration of the continuous magnitude it was supposed to measure—and this separation marked the history of thought as long as the “intuitive” concept of number remained dominant, i.e. nearly until the theoretical acknowledgement of complex numbers at the end of the 18th century.

Curiously enough, the same thinker who put such a strong emphasis on the separation between the realms of measures offered his successors some enigmatic hints to follow in order to overcome it and conceive of a kind of “universal mathematics” the possibility of which seemed to be excluded.

One might indeed raise the question whether first philosophy is universal, or deals with one genus, i.e. some one kind of being; for not even the mathematical sciences are all alike in this respect—geometry and astronomy deal with a certain particular kind of thing, while universal mathematics applies alike to all. 56

53 An. Post. 76a39.
54 Phys. 207b12.
55 See mainly Elem. Prop. VII.11–19.
56 Met. 1022a22–26. The universal mathematics is mentioned only twice in Aristotle’s works, with respect to the legislation of the first philosophy and with respect to the subject or content of demonstration (An. Post a17–25).
What is the subject of the universal mathematical theory? The response is rather complex: from Proclus to Russell, it nearly coincides with the conceptual development of mathematics. Still, two observations ought to be made here.

First, universal mathematics, as a theory comprising arithmetic and geometry, is supposed to found, unify and justify both the numerical and the geometrical measure, or, what means the same, to provide their common measure (whatever it may consist in). Thus, universal mathematics is a science of measure as such—the universal common measure sought after by Renaissance thought. Second, since the subject of universal mathematics, quantity, is supposed to consist in a pure relation, numbers and magnitudes may be construed as different symbolical expressions of the same universal intellectual matter, susceptible of boundless multiplicity of—indeed, yet related—symbolizations, formalizations and articulations. Hence, several consequences of capital importance follow.

First of all, the “intuitive” concept of number gets rid of its natural predominance. Ultimately, it represents an exceptional significant case of relational structure with since a “conceived relation is a real relation”. Number and magnitude are nothing more than specific articulations of a more general realm of relation, that might produce symbolic creations of various shapes and properties. Accordingly, negatives, irrationals and imaginaries may well be all numbers with no less claim to reality as compared to integers. More importantly, nothing prevents us from conceiving of even imaginary numbers of different kinds, insofar as they are based on a rule and follow from real relations. Mathematical liberty is a characteristic feature of universal mathematics.

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57 See Seidengart 2000, p. 66.
58 In general, there is no (Archimedean) quantity, neither discrete nor continuous, without an exact measure. For the case of numbers, see note 50. In the case of continuous magnitude, it is a ratio, ultimately a relation with a proper part of the magnitude (Elem. Def. V.1), and thus a comparison given by the possibility of equality (Def. V.5), that defines the notion of the kind of quantity (Def. V.3): “Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another” (Def. V.4). See also Elem. Def VII.3–5. Thus, the operational aspect is inherent to the nature of magnitude, but at the same time relies substantially on that of multiple, i.e. repeated addition of integers. This way, arithmetic is the first mathematical science; see Met. 982a20–28.
59 The relation is the only category that finally does not diminish its reality by being thought: “Sola relatio non habet ex hoc quod est huiusmodi quod aliquid ponat in rerum natura (...) unde quedam inveniuntur relationes quae nihil in rerum natura ponunt, sed in relatione tantum” (Thomas Aquinas, De veritate I, 5, ad 15). As the reality of relation is the same in human mind, in nature as well as in divine reason, it is precisely universal mathematics that makes man the “knot of the Universe” (see note 46) and nature, in turn, the “ladder to God”. Since, at the same time, the idea of universal mathematics leads directly to the notion of the (mathematical) infinite, it also grants the man an exclusive relation to God. Hence follows the infinite dignity of the Renaissance man.
60 Cf. the often-quoted sentence from Cantor’s Grundlagen paradoxically putting together seemingly contradictory categories—“for the essence of mathematics
Once number and magnitude share a common symbolic nature, nothing besides the force of habit precludes the unit from being divisible in the same manner as a length—and the other way round, a length from being considered as a unit. The analogy between arithmetic and geometry thus becomes primordial; it allows us to think the continuity instead of separation. The mimesis between the continuous and the discrete gives us, in a strange echo of Aristotle’s remark, a means to conceive the subtlety of their boundary relations, as if magnitudes were numbers and numbers were magnitudes, and to forge amphibian operational concepts like, for instance, the unit length mirroring numbers in the realm of magnitude.

Just as arithmetic consists of only four or five operations, namely addition, subtraction, multiplication, division, and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily (...).

However, an opposite, in a sense more radical path to overcome the gap is finally open: given that numbers and magnitudes are supposed to be made from the same matter, magnitudes may be mirrored in the realm of numbers as well. This form of expression provides us with an exclusive access to the orders of the infinite divisibility and possibility of quantity as such: through the medium of number, a direct relation between numbers and magnitudes can be analysed. Numbers and magnitudes are no longer separated, they are infinitely different. For, they are in the same (dis)proportion now as finite to infinite. A subtle network of boundary concepts arise: minimum, element, horizon, approximation, limit, unity and equality—all of them will find their place in the transcendent analysis. This path of universal mathematics leads directly to the labyrinth of the continuum and gives us an idea as to the way that the notion of (mathematical) infinity provides the key to the symbolic nature of the Renaissance. Finally, it is no wonder that, before obtaining their citizenship in the realm of numbers, the imaginaries will find the ultimate support for their full-fledged existence in the most advanced formal theory of infinitesimal calculus.

61 Cf. note 53.
62 See note 48.
63 Descartes 1954, p. 2.
64 This way, as we shall briefly touch later, was famously explored by Simon Stevin.
65 See especially A VI, 3, pp. 529–571.
66 We may observe that Cauchy, the author of substantial contributions to complex analysis, underwent an analogical experience as Cardano with respect to the status, dignity and power of “imaginary expressions”. From a certain reticence towards the imaginaries as mere abbreviations reducible in principle to relations between real numbers (Cauchy 1821, p. 173), the utility and universality of complex numbers drove him to their
RENAISSANCE OF NUMBERS
AND THREE CONTINUITIES

The “inveterate” idea of universal mathematics was reborn in the Renaissance era—in the same period as the theory of perspective and geometric algebra appeared. It is no coincidence, for both perspective and algebra consist in a sort of regular symbolical procedure. In the simplest case, perspective is a sort of equation formed from a certain point of view between the three-dimensional real world and the two-dimensional image on a plane; in algebra, the translation between the level of geometrical diagrams and that of “things” is even more nested. What is common to them is the continuity of symbolic expression articulated according to the geometric continuity: not only is the representation supposed to be substituted for the reality it depicts, but also different representations admit, as a rule, continuous transformations of one representation to another. This means that the continuity belongs to the underlying notion of the point of view. Similarly, there must be a fundamental continuity between symbolic expressions of known things and the unknown one, all the more if these symbolic representations are conceived of as variables, in order to be equated and transformed into the result.

Certainly, true subtlety is to be found in natural phenomena and in history in general. Nevertheless, the basic conceptual scheme presented above may help us to pinpoint at least some of important steps on the journey to the final admission of imaginary quantities. This journey, as we have already stated, consists in establishing the continuity between the real and the symbolic. For this purpose, the nature of things had to be enveloped into more complex and general symbolic realm of relation, in what we called the “symbolic turn”. At the same time, numbers and magnitudes had to be conceived of as symbolic expressions of the universal subject of quantity. For, in order to equate a true number to a fictitious one, the geometrical justification of algebra must be modified as well as the intuitive concept of number based on the indivisible unit. The number must become symbolic, operational, ambivalent, so that the other symbolic entities might spring forth. We shall briefly distinguish three continuities that must be established in order to conceive of the number as symbolic.

What are these symbolic milestones? Beside the aforementioned arts of perspective and algebra, it was the development of commercial activities, especially international ones, banking, calculations of interest or exchange rates and numerical practice in general (including

“naturalization”. First, in the form of a new mathematical object between formalist and intuitive stance, whose reality is based on “algebraic equivalences” (classes given by polynomial divisibility); and finally, in the form of “geometric quantity” (vector) comprising as a special case “algebraic” quantity (real number).


70 See note 59.
rational and negative numbers),\textsuperscript{71} that were constantly eroding the indivisible nature of the unit—after all, perspective and algebra were arts in the first place, i.e. disciplines intended to solve problems. We cannot here go into detail. No matter how substantial the concrete mathematical practice might be, in this article we shall take the continuity between theory and practice for granted. More importantly, since every number must be named or symbolized in order to be applied, the question of practice is to some extent reflected in the matter of mathematical symbolism, more accessible to our analysis. This is where the first continuity resides.

The gradual constitution of the decimal positional number system in arithmetic, “the art of tenths”,\textsuperscript{72} represents a major step towards the invention of the symbolic character of number. It relies on what we might call the most fundamental continuity of reference, since its expressive power surpasses a direct reference to the numbered thing to such an extent that it includes a boundary reference to nought. Not only does it provide means for manipulating huge numbers, completely dissociated from any evidence based on objects whatsoever,\textsuperscript{73} but it also establishes a continuity between symbols and symbolized number. Due to the former, the direct correspondence of the unit with the intuition of an object is increasingly blurred and finally replaced by analogy or by transmission of relations. Instead of individuals, numbers alone are taken into account. According to the latter, the fact that the name stands for numbers as well as for arithmetical operations allows us to reason and establish rules (rhetorical figures in fact) with numerals alone and without any recourse to numbers (let alone individual objects). A number, no matter how large, is always applicable to individuals and it is, for instance, always divisible by 3 if the sum of digits is divisible by 3.\textsuperscript{74}

The positional system plays the key role in the second continuity we shall treat here. It is the continuity of inversion underlying Stevin’s “geometrical” redefinition of number.\textsuperscript{75} This principle opens the path to universal mathematics from numbers towards magnitudes. Two points must be taken into account here. First, as we have just explained, the ambiguity of the symbolism’s reference to objects as results of operations, exempted numbers from the direct correspondence with objects, making them in a sense prior to counted individual things. Second, we already know that the Greeks encompassed the variety of heterogeneous magnitudes in the theory of proportions, and this was also the only

\textsuperscript{71} Guaquinta 2010, pp. 176–177. See note 31.

\textsuperscript{72} See mainly Stevin 1608, English translation of 1585 Flemish De thiende which presents the Renaissance development of the decimal notation.

\textsuperscript{73} This is the case of Archimedes’s Sand Reckoner which offers both, the construction of a number surpassing the horizon of things and the decimal positional notation system that enables it.

\textsuperscript{74} These rhetorical manipulations are the heart of the calculating approach to mathematics mentioned above (see note 19).

means to consider fractions or quantities less than one.\textsuperscript{76} These two ideas represent the elements of the new mathematical unity based on the infinite divisibility of the continuum. This is the “first matter” of quantity articulated by number. Once continuous magnitude is arithmetized, i.e. every length may be expressed by a number, the “geometrical number” emerges: “number is that expresseth the quantitie of each thing.”\textsuperscript{77}

Since numbers reside in an algorithmic procedure and do not depend on individuals anymore, no ontological prerogative follows from the correspondence of the unit with an individual thing. Once the indivisibility of the unit has lost its intuitive ground, the unit becomes a part of a number—and consequently, “the unit is a number”\textsuperscript{78} and every ratio between numbers is also a number: a fractional number.\textsuperscript{79}

If the numeral combinatorial system as a whole, by the continuity of inversion, is mirrored in every single of its units or elements, the richness of the realm of number replicates ad infinitum. Thus, the inexhaustible numerical diversity expressed by the decimal positional system may enter the unit itself. Number is expressed by a unique, possibly infinite decimal expansion attached to any magnitude, or as Stevin puts it: “as unity is the number by which we say that the quantity of an explained thing is one: and two, by which it is called two: and half, by which it is called half.”\textsuperscript{80}

Instead of the simple and indivisible unit, the principle of measure is simply geometrical number, or geometrical number as such.\textsuperscript{81} Number is to magnitude what humidity is to water, number is a continuous quantity.

As it penetrates like this into every part of its magnitude; just as a continuous wetness corresponds to a continuous body of water, a continuous magnitude corresponds to a continuous number. Likewise, as the continuous wetness of the entire body of water is subject to the same division and separation as the water, so the continuous number is subject to the same division and separation as its magnitude, in such a way that these two quantities cannot be distinguished by continuity or discontinuity.\textsuperscript{82}

\textsuperscript{76} See note 55.
\textsuperscript{77} Stevin 1608, p. A3.
\textsuperscript{78} Stevin 1958, pp. 495–496.
\textsuperscript{79} Stevin 1958, pp. 506–507.
\textsuperscript{80} Stevin 1958, p. 494.
\textsuperscript{81} Hence the ultimate question of the Renaissance: is there a universal common measure (see note 57)? What is the common measure to any quantity, the “possesst” of quantity? As a principle of every possible equality, it has to “enfold” every common measure and express the impossible proportion between finite and infinite. This is possible only in a rhetorical manner, or as Leibniz says: “fictione quadam possumus concipere, omnes quantitates homogeneas esse velut commensurabiles inter se, fingendo scilicet elementum aliquod infinite parvum” (GM VII, p. 39).
\textsuperscript{82} Stevin 1958, p. 494; cf. Klein 1968, p. 150.
What exactly does the “principle” mean? Stevin’s ground-breaking originality is revealed by the fact that he raised this question. With respect to universal mathematics, it may be rephrased as follows: what in the concept of number makes “the community and similarity of magnitude and number so universal that it almost resembles identity”?83 Now, within extension, the “manifest principle” is the point. Does the unit correspond to point?

We are approaching the decisive step where the concept of number is defined in terms of analogy based on more general mathematical reasons, i.e. as a symbolic entity resulting from a “community” of operational rules:

Just as a point is an adjunct to a line and not in itself a line, so is 0 an adjunct to number, and not a number itself.
Just as a point cannot be divided into parts, so 0 cannot be divided into parts.
Just as many points, yea an infinity of them, do not make a line, so many 0’s, even an infinity of them, do not form a number.84

The analogy taken from universal mathematics obliges us to pose 0 as the principle (i.e. beginning, commencement) of number. As a result, the theory of proportions is no longer the foundation of mathematics. It is replaced by much more intricate non-Archimedean determinations of quantity belonging to the labyrinth of the continuum: the rules of infinity (in particular with regard to the decimal expansion), conditions of equality, continuity between number and the principle of number or between a quantity and its limit, briefly conditions of continuity as such. Nevertheless, much is gained at the end of the day: besides the notational system suitable for extensions and generalisations,85 we are reaching the level of quantity, the first matter of universal mathematics and unlimited source of relations to be symbolized and articulated. As an “intensification” of the continuity of reference, the “microcosmic” continuity of inversion guaranteed the symmetrical validity of arithmetical operations. As a result, all quantities become homogeneous. Since the concept of number is now promoted at the level of the measure of any quantity, it may be symbolised universally (as unknown as well as indeterminate), and thus give rise to the new formal order of relations—relations between forms, expressed by “blind” formulas continuously substituted for quantities.

Here we set foot on the other path of universal mathematics, the symbolic

83 Stevin 1585, p. 498.
84 Stevin 1585 [1958], p. 498 [43].
85 First of all, (fractional) exponents or differential orders. Stevin himself conceives a more general notion of “algebraic number”, i.e. indeterminate quantity supplied with an exponent or “dignity” (Stevin 1585, pp. 518–519) and even “algebraic multinomials” (Stevin 1585, p. 521), polynomials considered as mathematical objects independent of the theory of equations (Rabouin 2009, p. 242).
The Renaissance of Numbers: On Continuity, Nature of Complex Numbers and the Symbolic Turn

Jan Makovský

Path of algebra. It was foreshadowed, as we already know, by the admission of false numbers among true ones; and from the false numbers, it went almost straightforwardly to the “impossible” ones. The algebraic approach to universal mathematics establishes a formal unity of mathematics. Accordingly, algebra is not subject to justification anymore, but it is subject to the interpretation of the symbolic representation (species) within the matter of number or magnitude. This is also the reason why it makes visible and regular heterogeneities between them, most notably the law of dimensional homogeneity, and brings into existence symbolic operational creations like, for instance, a figure of dimension higher than three. The convergence of both paths is the unity of universal mathematics: while the algebraic path leads to a formal system of abstract magnitudes defined by the operational rules they satisfy, providing a boundless liberty to create new objects, the labyrinthian path to universal mathematics guarantees the “matter”, ultimate reality or meaningfulness of its uniform quantitative base. But only both together lead to an intelligible justification of complex numbers. The symbolic continuity between zero and positive quantity is necessary in order to extend the intelligibility to the negatives, while, as we have shown, the $\sqrt{-1}$ is certainly not a part of 1 (or even of $-1$) made of the same “matter”.

The mirror image of possibility itself which “exhibits cases of Problems that are impossible as if they were possible”, is inconceivable without the universal expressive and analytical power of the formal system. The first example of such a system is found in François Viète’s Isagoge, since it was Viète who following Diophantus’s use of unknowns in the Arithmetics, represented the ambiguous nature of a general number or a number as magnitude by a letteral sign. Thus, analysis “no longer limits its reasoning to numbers, a shortcoming of the old analysts, but works with a newly discovered symbolic logistic.”

From the historical point of view, it is worthwhile to note that until the time of van Roomen and Descartes in particular, the symbolic algebra was not identified or even brought together with the idea of universal mathematics, see Rabouin 2009, pp. 237–239.

Due to the opacity or under-determination of symbols, external in fact to objects they express. See note 7.


For the axiomatics, see Viète 1616, pp. 1–2.

From Newton 1728, p. 195. Here, as well as in Leibniz (GP I, pp. 405–406; GM VII, pp. 73–76), for instance, the square root of negative number indicates a case in which there is no intersection, i.e. the “impossible intersection”, between a line and a circle—instead of one or two intersections expressed by real roots. Accordingly, by means of the complex numbers, the universal situation of the two figures in a plane is grasped, corresponding to the totality of solutions of the equation (see note 3).

See Bos 2001, p. 146; Klein 1968, pp. 95–143.

See note 87.

Viète 2006, p. 13. Most notably, Viète introduced letteral notation of coefficients. Consequently, placing them on
The symbolic purity of analysis free of rhetorical elements was “restored” by Thomas Harriot, Viète’s successor, who generalized the notion of equality by allowing equation of a term or a “thing” to zero or nothing. This step is more serious than it seems from today’s point of view. Equating all terms of equation to zero, on the one hand, pushes the meaning of the equation to the purely symbolic level, allowing the focus to be limited on its “canonical”, i.e. polynomial structure. On the other hand, it lets us consider the sign $-$, for instance, ambiguously as an operation as well as an object and formally incorporate the “impossible” object (a case, in fact) into the order of the “uniform and continuous” application of rules. Here, the two paths of universal mathematics intersect and provide a starting point from which it is not only possible to move towards the code of the Cartesian algebra—but also, for the reasons exposed above, towards the subtlety of infinitesimal analysis. The “impossible” roots first turn into “inexplicable” (i.e. un-related to coefficients) ones and then into “imaginary” quantities, before they finally become complex numbers as we know them.

In truth, the nature of things, mother of eternal varieties, or rather the Divine Reason, holds fast to its splendid variety. Thus, it does not let
all things be confined within one and only kind. Therefore, an elegant and admirable way out is found in that miracle of Analysis, that portent of the ideal world, almost Amphibian between Being and non-Being, that we call an imaginary root.\footnote{Verum enim tenacior est varietatis suae pulcherrimae Natura rerum, aeternarum varietatum parens, vel potius Divina Mens, quam ut omnia sub unum genus compingi patiatur. Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibia, quod radicum idealem appellamus” (GM V, p. 357).}

Besides “real” quantities, such an infinitely prodigal nature can certainly bear the expense of accommodating all sorts of fictitious, imaginary creations such as infinitesimals, infinitely great numbers, horned angles or osculating circles.\footnote{See GM IV, p. 92.} For, they all stem from the same source, while their mutual differences consist, precisely, in the complexity of the relations they express. Since they are all symbolic and continuous, any distinction between fiction and reality based on intuition or some kind of phenomenological evidence is deceptive a priori.

How then, can we tell truth from mere fiction? Is the symbolic interplay enough to provide a mathematical object, a curve for example, with its “true nature”? These are the questions that the mathematics as well as the philosophy and literature of the classical era all dealt with simultaneously, all the more that the mathematical demonstration traditionally represented a standard of certitude and truth.\footnote{See the famous anecdote of the libertine age given by Molière’s Don Juan, III.1.} Be it Hobbes-Wallis controversy or Molyneux’s problem, we must reserve them for some future treatise. What matters to us at this moment is this simple consequence: by the same reasoning, such an infinitely variable nature might admit the imaginary roots of different kinds, as though there are different orders of the infinitely small.\footnote{This, in fact was the conviction of Leibniz who claimed that there was an infinite hierarchy of ever more complex kinds of imaginary roots, \(\sqrt{-1}\) being different kind than \(\sqrt{\sqrt{-1}}\). This view stems from an error committed in decomposition of the polynomial \(x^4 + a^4\) undertaken as a part of a general linearization attempt to find a method of integrating rational fractions. (GM V, pp. 350–361). We can sense the connection of the mistake with Leibniz’s general metaphysical conviction expressed above (note 98).} While algebraic operations, even with purely symbolic numbers such as exponents or coefficients, can be traced back to real quantities, imaginary ones rely entirely on analogy. But isn’t it the same analogy that admits an unlimited number of iterations, combinations and complications, giving birth to an obscure crowd of objects more and more detached from any significance whatsoever? Among other things, such a state of affairs would have disastrous consequences for the (general) validity of the fundamental theorem of algebra.

We have claimed that the nature of complex numbers arises through the convergence of the two main approaches towards the subject of mathematics, demonstration and calculation. Consequently, these are the two
main strategies of the justification of imaginaries. In the absence of the “intuitive” geometric justification, the definitive question is whether the analogy is complete—or, in other words, whether there are no other kinds of imaginary roots (quantities), i.e. the symbol \( a + b\sqrt{-1} \) is stable in ordinary operations of algebra.

The demonstration of stability is the last of the symbolic milestones mentioned in this brief essay. This is because such a stability is a strong indication, constructed almost entirely on the symbolic base of algebra, that the expressions \( a + b\sqrt{-1} \) may be well treated as a kind of numbers. It is also, as we have just remarked, a necessary condition for the validity of the fundamental theorem of algebra. The theorem appears in d’Alembert’s *Reflexions sur la cause Générale des vents*.

It is certain that every algebraic quantity, be it composed of any imaginary quantities we wish, can always be reduced on \( A + B\sqrt{-1} \), \( A \) and \( B \) being real quantities. Hence, if given quantity is to be real, it follows that \( B = 0 \).

The proof has four steps, corresponding to the ordinary operations of algebra. First it states for division

\[
\frac{a + b\sqrt{-1}}{g + h\sqrt{-1}} = A + B\sqrt{-1}
\]

The case is clear. By simple algebraic procedures, we isolate \( A \) and \( B \) expressed by means of the original real quantities. Second, for raising to power

\[
a + b\sqrt{-1}^{m+n\sqrt{-1}} = A + B\sqrt{-1}
\]

Since the last case (addition, subtraction, multiplication) is obvious, this is the key point of the whole demonstration. It is worth noting that, instead of sorting out expressions according to their complexity, d’Alembert takes as a starting point the general expression. Since the proof is as sketchy as it is technical, at the same time as it involves several deficiencies, we will not reproduce it as it stands. In nuce, it consists of taking the logarithm of the equation and differentiating it. Thus we obtain expressions of \( A \) and \( B \) having the form of cosines and sines supplied with a module and an argument representing a complex number. Instead, we will limit ourselves here to general remarks in accordance with the main purpose of this paper.

First and foremost, the proof relies on the idea of the differential, i.e. the continuous variation of the symbolic expression, that concentrates and incorporates all the continuities we have treated above. Moreover, only by virtue of all of them does a syntactical
demonstration of universality through the stability of signs makes any sense—only on this level of symbolic abstraction does autonomy mean existence. Second, the proof is not purely formal. It is carried out with the help of trigonometric representation of imaginary quantities, manifesting the traces of geometric intuition concealed within the continuity of symbolic expression.

CONCLUDING REMARKS
We began this paper with the intuitive justification of impossible entities that do not answer neither the question about “how much” nor about “how many”, but symbolically comprise answers to both questions and even more. We are ending it at the same place, having spent all of our time explaining the conditions of that “intuition” or, to say it with Gauss, how a “difference in itself” can be fully determinate “although we can communicate our intuition of this difference to others only by appeal to really existent material things”. The answer, as we intended to show, lies in the autonomy of symbolism conceivable on the assumption that nature itself has turned symbolic. Only the nature of the symbol is able to reconcile limitations of things with their infinite principle manifested through the fundamental continuity, an inexhaustible source of every possible proportion. Due to the fundamental continuity, thus, it is relation, analogy, order and harmony that count and that even constitute the ambivalent symbolic “bodies” of things. Once the individuality of a thing may be given by an operation, its nature become universal and will always find a correspondence in other individuals—a correspondence that can be reflected, symbolized and articulated. This holds pre-eminently for the symbols of number and magnitude and the sciences of arithmetic and geometry intersecting in what is called universal mathematics, since the geometrical continuity represents the most significant image of the fundamental continuity accessible only by means of ambiguous symbols. And that is also the reason why it is here, in the symbolic turn of the Renaissance, where we find, among other symbolic creations, the liberty of conceiving of such an enigmatic notion as imaginary quantity. It is in the boundless mathematical liberty reliant on the symbolic nature of the Renaissance and the rich ambivalence of universal mathematics where the history of complex numbers begins. It is a history of subtlety and subtlety is also what it can teach us.

109 Gauss 1876, p. 178; tr. Ewald 2005a, p. 313. Here, Gauss appended an important footnote: “Kant already made both observations, but one cannot understand how this acute philosopher could have believed that the first gives a proof of his opinion that space is only the form of our outer intuition, since the second so clearly proves the opposite, namely, that space must have a real significance independent of our mode of intuition”. 
ABBREVIATIONS

Aristotle
An. Post. Analytica posteriora
Met. Metaphysica
Phys. Physica

Descartes, R.

Euclid
Elem. Elementa

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